Supplementary Material

Proof of Lemma 1

We first introduce the following two lemmas.

Lemma 5. *(Lemma 6 in Hazan and Minasyan (2020)) Under Assumption 1, the linear value oracle $\mathcal{M}_{\mathcal{K}}(\cdot)$ is convex and D-Lipschitz, i.e.,*

$$\forall \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^d,\ |\mathcal{M}_{\mathcal{K}}(\mathbf{y}_1) - \mathcal{M}_{\mathcal{K}}(\mathbf{y}_2)| \leq D\|\mathbf{y}_1 - \mathbf{y}_2\|_2. \tag{38}$$

Lemma 6. *(Lemma 11 in Hazan and Minasyan (2020)) The function $h^*_\eta(\mathbf{y}) = \mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\mathcal{M}_{\mathcal{K}}(\mathbf{y} + \frac{1}{\eta}\cdot\mathbf{v})\right]$ is ηdD-smooth, given $\mathcal{M}_{\mathcal{K}}(\cdot) : \mathbb{R}^d \to \mathbb{R}$ is D-Lipschitz, i.e., $\forall \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^d$*

$$h^*_\eta(\mathbf{y}_1) \leq h^*_\eta(\mathbf{y}_2) + \left\langle \nabla h^*_\eta(\mathbf{y}_2), \mathbf{y}_1 - \mathbf{y}_2\right\rangle + \frac{\eta dD}{2}\|\mathbf{y}_1 - \mathbf{y}_2\|_2^2. \tag{39}$$

Lemma 6 implies that $h^*_\eta(\cdot)$ is ηdD-smooth. Therefore, we have

$$\forall \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^d, \|\nabla h^*_\eta(\mathbf{y}_1) - \nabla h^*_\eta(\mathbf{y}_2)\|_2 \leq \eta dD\|\mathbf{y}_1 - \mathbf{y}_2\|_2. \tag{40}$$

Assumption 3 indicates that communications between local learners in D-OCO are modeled via a doubly stochastic matrix P. Let $\bar{\mathbf{z}}_t = \frac{1}{n}\sum_{j=1}^n \mathbf{z}_{t,j}$ be the average of the dual variables for all learners at round t. By exploiting the special properties of P, we can upper bound the difference between $\bar{\mathbf{z}}_t$ and $\mathbf{z}_{t,i}$ for any local learner i at round t, as shown below.

Lemma 7. *(Lemma 6 in Zhang et al. (2017)) Let $\bar{\mathbf{z}}_t = \frac{1}{n}\sum_{j=1}^n \mathbf{z}_{t,j}$ and $\mathbf{z}_{t,i} = \sum_{j\in N_i} P_{ij}\mathbf{z}_{t-1,j} + \mathbf{u}$, where $\mathbf{u}$ is a vector and $\|\mathbf{u}\|_2 \leq G$ .Under Assumption 3, for any learner $i \in V$ at round t*

$$\|\mathbf{z}_{t,i} - \bar{\mathbf{z}}_t\|_2 \leq \frac{\sqrt{n}G}{1-\sigma_2(P)}, \tag{41}$$

where $\sigma_2(P)$ is the second largest eigenvalue of the communication matrix P.

Let $\mathbf{z}_{t,i}$ and $\mathbf{x}_{t,i}$ be defined as that in Algorithm 1. Denote $\bar{\mathbf{z}}_{t-1} = \frac{1}{n}\sum_{j=1}^n \mathbf{z}_{t-1,j}$ and $\bar{\mathbf{x}}_t = \nabla h^*_\eta(-\bar{\mathbf{z}}_{t-1})$, then we have

$$\begin{aligned}\|\bar{\mathbf{x}}_t - \mathbf{x}_{t,i}\|_2 &= \|\nabla h^*_\eta(-\bar{\mathbf{z}}_{t-1}) - \nabla h^*_\eta(-\mathbf{z}_{t-1,i})\|_2\\ &\overset{(40)}{\leq} \eta dD\|\mathbf{z}_{t-1,i} - \bar{\mathbf{z}}_{t-1}\|_2\\ &\overset{(41)}{\leq} \eta dD\frac{\sqrt{n}G}{1-\sigma_2(P)} = \epsilon,\end{aligned} \tag{42}$$

Hence, we have proved Lemma 1.

Proof of Lemma 2

Let $\bar{\mathbf{x}}_t = \nabla h^*_\eta(-\bar{\mathbf{z}}_{t-1})$, $\epsilon = \eta dD\frac{\sqrt{n}G}{1-\sigma_2(P)}$ and $\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x}\in\mathcal{K}}\sum_{j=1}^n\sum_{t=1}^T f_{t,j}(\mathbf{x})$. Under Assumption 2, by using the convexity of $f_{t,j}(\mathbf{x})$ and triangle inequality, we have

$$\begin{aligned}f_{t,j}(\mathbf{x}_{t,i}) &\leq f_{t,j}(\bar{\mathbf{x}}_t) + \langle\nabla f_{t,j}(\mathbf{x}_{t,i}), \mathbf{x}_{t,i} - \bar{\mathbf{x}}_t\rangle\\ &\leq f_{t,j}(\bar{\mathbf{x}}_t) + G\|\mathbf{x}_{t,i} - \bar{\mathbf{x}}_t\|_2\end{aligned} \tag{43}$$

$$\begin{aligned}f_{t,j}(\bar{\mathbf{x}}_t) &\leq f_{t,j}(\mathbf{x}_{t,j}) + \langle\nabla f_{t,j}(\bar{\mathbf{x}}_t), \mathbf{x}_{t,j} - \bar{\mathbf{x}}_t\rangle\\ &\leq f_{t,j}(\mathbf{x}_{t,j}) + G\|\mathbf{x}_{t,j} - \bar{\mathbf{x}}_t\|_2\end{aligned} \tag{44}$$

Then,using Lemma 1, (43) and (44), we have

$$
\begin{aligned}
\sum_{t=1}^{T}[f_{t,j}(\mathbf{x}_{t,i})-f_{t,j}(\mathbf{x}^*)] &\overset{(43)}{\leq} \sum_{t=1}^{T}[f_{t,j}(\bar{\mathbf{x}}_t)+G\|\mathbf{x}_{t,i}-\bar{\mathbf{x}}_t\|_2-f_{t,j}(\mathbf{x}^*)] \\
&\overset{(42),(44)}{\leq} \sum_{t=1}^{T}[f_{t,j}(\mathbf{x}_{t,j})+G\|\mathbf{x}_{t,j}-\bar{\mathbf{x}}_t\|_2-f_{t,j}(\mathbf{x}^*)]+\epsilon GT \\
&\overset{(42)}{\leq} \sum_{t=1}^{T}[f_{t,j}(\mathbf{x}_{t,j})-f_{t,j}(\mathbf{x}^*)]+2\epsilon GT \\
&\leq \sum_{t=1}^{T}\langle\nabla_{t,j},\mathbf{x}_{t,j}-\mathbf{x}^*\rangle+2\epsilon GT \\
&= \sum_{t=1}^{T}[\langle\nabla_{t,j},\mathbf{x}_{t,j}-\bar{\mathbf{x}}_t\rangle+\langle\nabla_{t,j},\bar{\mathbf{x}}_t-\mathbf{x}^*\rangle]+2\epsilon GT \\
&\leq \sum_{t=1}^{T}[G\|\mathbf{x}_{t,j}-\bar{\mathbf{x}}_t\|_2+\langle\nabla_{t,j},\bar{\mathbf{x}}_t-\mathbf{x}^*\rangle]+2\epsilon GT \\
&\overset{(42)}{\leq} \sum_{t=1}^{T}\langle\nabla_{t,j},\bar{\mathbf{x}}_t-\mathbf{x}^*\rangle+3\epsilon GT.
\end{aligned} \tag{45}
$$

Summing up both side of (45) from $j=1$ to n, we have

$$
\mathrm{Regret}_i=\sum_{j=1}^{n}\sum_{t=1}^{T}[f_{t,j}(\mathbf{x}_{t,i})-f_{t,j}(\mathbf{x}^*)]\leq\sum_{j=1}^{n}\sum_{t=1}^{T}\langle\nabla_{t,j},\bar{\mathbf{x}}_t-\mathbf{x}^*\rangle+3\epsilon GTn\leq n\sum_{t=1}^{T}\left\langle\bar{\nabla}_t,\bar{\mathbf{x}}_t-\mathbf{x}^*\right\rangle+3\epsilon GTn, \tag{46}
$$

in which $\bar{\nabla}_t=\frac{1}{n}\sum_{j=1}^{n}\nabla_{t,j}$.

Proof of Lemma 3

Lemma 8. *For any $\mathbf{v}\sim\mathbb{B}$, $h_\eta(\mathbf{x})$ is upper bounded by $\frac{D}{\eta}$ under Assumption 1, i.e.,*

$$
\forall\mathbf{x}\in\mathcal{K}, h_\eta(\mathbf{x})\leq D/\eta. \tag{47}
$$

By applying weak duality and Lemma 8, we have

$$
D(\bar{\lambda}_1^*,\cdots,\bar{\lambda}_T^*)\leq\min_{\mathbf{x}\in\mathcal{K}}\{h_\eta(\mathbf{x})+\sum_{t=1}^{T}F_t(\mathbf{x})\}\leq\max_{\mathbf{x}\in\mathcal{K}}h_\eta(\mathbf{x})+\min_{\mathbf{x}\in\mathcal{K}}\sum_{t=1}^{T}F_t(\mathbf{x})\leq\frac{D}{\eta}+\min_{\mathbf{x}\in\mathcal{K}}\sum_{t=1}^{T}F_t(\mathbf{x}). \tag{48}
$$

Proof of Lemma 8

(The following proof can also be found in Hazan and Minasyan (2020). Here, we present it in detail).

First, we recall that $h_\eta^*(\mathbf{y})=\mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\mathcal{M}_\mathcal{K}(\mathbf{y}+\frac{1}{\eta}\cdot\mathbf{v})\right]$. Then, under Assumption 1, we have $\forall\mathbf{x}\in\mathcal{K},\mathbf{y}\in\mathbb{R}^d$,

$$
\begin{aligned}
\langle\mathbf{x},\mathbf{y}\rangle-h_\eta^*(\mathbf{y})=\langle\mathbf{x},\mathbf{y}\rangle-\mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\mathcal{M}_\mathcal{K}(\mathbf{y}+\frac{1}{\eta}\cdot\mathbf{v})\right]&=\mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\langle\mathbf{x},\mathbf{y}\rangle-\max_{\mathbf{x}'\in\mathcal{K}}\left\langle\mathbf{y}+\frac{1}{\eta}\cdot\mathbf{v},\mathbf{x}'\right\rangle\right] \\
\leq\mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\langle\mathbf{x},\mathbf{y}\rangle-\left\langle\mathbf{y}+\frac{1}{\eta}\cdot\mathbf{v},\mathbf{x}\right\rangle\right]&=\mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\left\langle-\frac{1}{\eta}\cdot\mathbf{v},\mathbf{x}\right\rangle\right] \\
\leq\mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\frac{\|\mathbf{v}\|_2\|\mathbf{x}\|_2}{\eta}\right]\leq\mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\frac{D}{\eta}\right]&=\frac{D}{\eta}.
\end{aligned} \tag{49}
$$

So we have

$$
h_\eta(\mathbf{x})=\langle\mathbf{x},\mathbf{y}\rangle-h_\eta^*(\mathbf{y})\leq D/\eta. \tag{50}
$$

Proof of Lemma 4

We first introduce the following two lemmas.

Lemma 9. *For any $\mathbf{v} \sim \mathbb{B}$, $h^*_\eta(0)$ is upper bounded by $\frac{D}{\eta}$ under Assumption 1, i.e.,*

$$h^*_\eta(0) \le D/\eta. \tag{51}$$

Lemma 10. *Let $\bar{\nabla}_t = \frac{1}{n}\sum_{j=1}^n \nabla_{t,j}$ and $\bar{\mathbf{z}}_t = \frac{1}{n}\sum_{j=1}^n \mathbf{z}_{t,j}$. Under Assumption 3 we have*

$$\bar{\mathbf{z}}_t = \bar{\mathbf{z}}_{t-1} + \bar{\nabla}_t, \tag{52}$$

Moreover, if $\mathbf{z}_{0,i} = \mathbf{0}$, there is $\bar{\mathbf{z}}_0 = \frac{1}{n}\sum_{j=1}^n \mathbf{z}_{0,j} = \mathbf{0}$ and we have $\bar{\nabla}_{1:t} = \bar{\mathbf{z}}_t$.

Then, we denote $\bar{\Delta}_t$ as the difference value of $D(\bar{\lambda}_1, \cdots, \bar{\lambda}_T)$ with two consecutive rounds:

$$\begin{aligned}
\bar{\Delta}_t &= D(\bar{\lambda}_1^t, \cdots, \bar{\lambda}_T^t) - D(\bar{\lambda}_1^{t-1}, \cdots, \bar{\lambda}_T^{t-1}) \\
&= D(\bar{\nabla}_1, \cdots, \bar{\nabla}_t, 0, \cdots, 0) - D(\bar{\nabla}_1, \cdots, \bar{\nabla}_{t-1}, 0, \cdots, 0) \\
&= -\left[h^*_\eta\left(-\bar{\nabla}_{1:t}\right) - h^*_\eta\left(-\bar{\nabla}_{1:t-1}\right)\right] - F^*_t(\bar{\nabla}_t) + F^*_t(0).
\end{aligned} \tag{53}$$

According to the definition of $\bar{\Delta}_t$, we have

$$\begin{aligned}
\bar{\Delta}_t &\overset{(53)}{=} -\left[h^*_\eta\left(-\bar{\nabla}_{1:t}\right) - h^*_\eta\left(-\bar{\nabla}_{1:t-1}\right)\right] - F^*_t(\bar{\nabla}_t) + F^*_t(0) \\
&\overset{(39)}{\ge} \left\langle \bar{\nabla}_t, \nabla h^*_\eta\left(-\bar{\nabla}_{1:t-1}\right)\right\rangle - \frac{\eta d D}{2}\|\bar{\nabla}_t\|_2^2 - F^*_t(\bar{\nabla}_t) + F^*_t(0) \\
&= \left\langle \bar{\nabla}_t, \bar{\mathbf{x}}_t\right\rangle - F^*_t(\bar{\nabla}_t) - \frac{\eta d D}{2}\|\bar{\nabla}_t\|_2^2 + F^*_t(0) \\
&\ge F_t(\bar{\mathbf{x}}_t) - \frac{\eta d D}{2}G^2 + F^*_t(0).
\end{aligned} \tag{54}$$

The first inequality is because $h^*_\eta(\mathbf{y})$ is $\eta d D$-smooth (Lemma 6). The second inequality is due to Assumption 2 and the Fenchel dual identity $F^*_t(\bar{\nabla}_t) = \left\langle \bar{\nabla}_t, \mathbf{x}\right\rangle - F_t(\mathbf{x}) = 0$ for the linear function $F_t(\mathbf{x}) = \left\langle \bar{\nabla}_t, \mathbf{x}\right\rangle$. The last equality is because $\bar{\mathbf{x}}_t = \nabla h^*_\eta\left(-\bar{\mathbf{z}}_{t-1}\right) = \nabla h^*_\eta\left(-\bar{\nabla}_{1:t-1}\right)$ according to Lemma 10. The inequality (54) can be simplified as follows:

$$\bar{\Delta}_t = D(\bar{\nabla}_1, \cdots, \bar{\nabla}_t, 0, \cdots, 0) - D(\bar{\nabla}_1, \cdots, \bar{\nabla}_{t-1}, 0, \cdots, 0) \ge F_t(\bar{\mathbf{x}}_t) - \frac{\eta d D}{2}G^2 + F^*_t(0). \tag{55}$$

By summing up (55) for all $t = 1, \cdots, T$, we have

$$\begin{aligned}
\sum_{t=1}^T \bar{\Delta}_t &= D(\bar{\nabla}_1, \cdots, \bar{\nabla}_T) - D(0, \cdots, 0) \\
&= D(\bar{\nabla}_1, \cdots, \bar{\nabla}_T) - \left(-h^*_\eta\left(0\right) - \sum_{t=1}^T F^*_t(0)\right) \\
&\ge \sum_{t=1}^T F_t(\bar{\mathbf{x}}_t) - \frac{\eta d D}{2}G^2 T + \sum_{t=1}^T F^*_t(0),
\end{aligned} \tag{56}$$

which further implies that

$$\begin{aligned}
D(\bar{\nabla}_1, \cdots, \bar{\nabla}_T) &\ge \sum_{t=1}^T F_t(\bar{\mathbf{x}}_t) - \frac{\eta d D}{2}G^2 T - h^*_\eta\left(0\right) \\
&\ge \sum_{t=1}^T F_t(\bar{\mathbf{x}}_t) - \frac{\eta d D}{2}G^2 T - \frac{D}{\eta},
\end{aligned} \tag{57}$$

where the last inequality is due to Lemma 9.

Proof of Lemma 9

(The following proof can also be found in Hazan and Minasyan (2020). Here, we present it in detail).
Since $\mathcal{M}_{\mathcal{K}}(0) = 0$, by Lipschitzness of $\mathcal{M}_{\mathcal{K}}(\cdot)$ (Lemma 5), we have

$$\left|\mathcal{M}_{\mathcal{K}}\left(\frac{1}{\eta}\cdot\mathbf{v}\right)\right| \le D\frac{\|\mathbf{v}\|_2}{\eta} \le \frac{D}{\eta}, \tag{58}$$

where $\mathbf{v}$ is sampled from an unit ball $\mathbb{B}$. So we have

$$h^*_\eta(0) = \mathbb{E}_{\mathbf{v}\sim\mathbb{B}}\left[\mathcal{M}_{\mathcal{K}}\left(\frac{1}{\eta}\cdot\mathbf{v}\right)\right] \le \frac{D}{\eta}. \tag{59}$$

Proof of Lemma 10

Let $\bar{\nabla}_t = \frac{1}{n}\sum_{i=1}^n \nabla_{t,i}$ and $\bar{\mathbf{z}}_t = \frac{1}{n}\sum_{i=1}^n \mathbf{z}_{t,i}$ where $\mathbf{z}_{t,i} = \sum_{j\in N_i} P_{ij}\mathbf{z}_{t-1,j} + \nabla_{t,i}$. Then, we have

$$\bar{\mathbf{z}}_t = \frac{1}{n}\sum_{i=1}^n \mathbf{z}_{t,i} = \frac{1}{n}\sum_{i=1}^n \left(\sum_{j\in N_i} P_{ij}\mathbf{z}_{t-1,j} + \nabla_{t,i}\right) = \frac{1}{n}\sum_{j=1}^n\sum_{i=1}^n P_{ij}\mathbf{z}_{t-1,j} + \frac{1}{n}\sum_{i=1}^n \nabla_{t,i} = \bar{\mathbf{z}}_{t-1} + \bar{\nabla}_t, \tag{60}$$

where the last equality is because Assumption 3 holds that $\sum_{j=1}^n P_{ij} = \sum_{j\in N_i} P_{ij}$ and $\sum_{i=1}^n P_{ij} = 1$. If $\mathbf{z}_{0,i} = \mathbf{0}$, there is $\bar{\mathbf{z}}_0 = \frac{1}{n}\sum_{j=1}^n \mathbf{z}_{0,j} = \mathbf{0}$ and we have

$$\bar{\nabla}_{1:t} = \sum_{r=1}^t \bar{\nabla}_r = \sum_{r=1}^t (\bar{\mathbf{z}}_r - \bar{\mathbf{z}}_{r-1}) = \bar{\mathbf{z}}_t. \tag{61}$$

Proof of Theorem 2

Proof of general convex losses

In Algorithm 2, all the random vectors are independent and identically distributed (i.i.d.), and sampled from an unit ball $\mathbb{B}$ uniformly. At round t, we denote $\xi_{t,i} = \left\{\mathbf{v}_{t,i}^1, \cdots, \mathbf{v}_{t,i}^m\right\}$ as the randomness of learner i. And the sample randomness is denoted as $\xi_t = \{\xi_{t,1}, \cdots, \xi_{t,n}\}$ at round t. For brevity, we denote the random variables until round t as $\xi_{[t]} = \{\xi_1, \cdots, \xi_t\}$ and correspondingly, for local learner i the random variables are denoted as $\xi_{[t],i} = \{\xi_{1,i}, \cdots, \xi_{t,i}\}$.

We first introduce following three lemmas.

Lemma 11. *(Lemma 15 in (Hazan and Minasyan 2020)) Let $Z_1, \cdots, Z_m \sim \mathcal{Z}$ be i.i.d. samples of a bounded random vector $Z \in \mathbb{R}^d$, $\|Z\|_2 \le D$, with mean $\bar{Z} = \mathbb{E}[Z]$. Denote $\bar{Z}_m = \frac{1}{m}\sum_{u=1}^m Z_u$, then $\mathbb{E}_{\mathcal{Z}}\left[\|\bar{Z}_m - \bar{Z}\|_2\right] \le \sqrt{\mathbb{E}_{\mathcal{Z}}\left[\|\bar{Z}_m - \bar{Z}\|_2^2\right]} \le \frac{2D}{\sqrt{m}}$.*

Lemma 12. *Define $\check{\mathbf{x}}_{t,i} = \nabla h_\eta^*(-\tilde{\mathbf{z}}_{t-1,i})$, $\hat{\mathbf{x}}_t = \nabla h_\eta^*(-\tilde{\mathbf{z}}_{t-1})$ where $\tilde{\mathbf{z}}_{t-1} = \frac{1}{n}\sum_{j=1}^n \tilde{\mathbf{z}}_{t-1,j}$ and $\tilde{\nabla}_{t,i}, \tilde{\mathbf{z}}_{t-1,i}$ are both defined in Algorithm 2 . Then we have*

$$\|\check{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\|_2 \le \epsilon, \tag{62}$$

where $\epsilon = \eta d D \frac{\sqrt{n}G}{1-\sigma_2(P)}$.

Lemma 13. *Let $\tilde{\mathbf{z}}_{t,j}$, $\tilde{\mathbf{x}}_{t,i}$ and $\tilde{\nabla}_{t,i}$ be defined as that in Algorithm 2. Define $\hat{\mathbf{x}}_t = \nabla h_\eta^*(-\tilde{\mathbf{z}}_{t-1})$ where $\tilde{\mathbf{z}}_{t-1} = \frac{1}{n}\sum_{j=1}^n \tilde{\mathbf{z}}_{t-1,j}$, then we have*

$$\mathbb{E}_{\xi_{[T]}}\left[\|\hat{\mathbf{x}}_t - \tilde{\mathbf{x}}_{t,i}\|_2\right] \le \epsilon + \frac{2D}{\sqrt{m}}, \tag{63}$$

where $\epsilon = \eta d D \frac{\sqrt{n}G}{1-\sigma_2(P)}$.

Remark 7. The expected action $\hat{\mathbf{x}}_t$ can be viewed as that played by a virtual centralized learner. Lemma 13 indicates that the distance between the expected action $\hat{\mathbf{x}}_t$ of the virtual learner and the sampling action $\tilde{\mathbf{x}}_{t,i}$ of learner i is upper bounded. In the following, by Lemma 13, we can convert the global regret analyze to this virtual one.

Define $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathcal{K}} \sum_{j=1}^n\sum_{t=1}^T f_{t,j}(\mathbf{x})$ and $\hat{\mathbf{x}}_t = \nabla h_\eta^*(-\tilde{\mathbf{z}}_{t-1})$ where $\tilde{\mathbf{z}}_{t-1} = \frac{1}{n}\sum_{j=1}^n \tilde{\mathbf{z}}_{t-1,j}$. By exploiting the convexity of $f_{t,j}(\mathbf{x})$ and triangle inequality, we have

$$\begin{aligned} f_{t,j}(\tilde{\mathbf{x}}_{t,i}) &\le f_{t,j}(\hat{\mathbf{x}}_t) + \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\rangle \\ &\le f_{t,j}(\hat{\mathbf{x}}_t) + G\|\tilde{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\|_2 \end{aligned} \tag{64}$$

$$\begin{aligned} f_{t,j}(\hat{\mathbf{x}}_t) &\le f_{t,j}(\tilde{\mathbf{x}}_{t,j}) + \langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \tilde{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_t\rangle \\ &\le f_{t,j}(\tilde{\mathbf{x}}_{t,j}) + G\|\tilde{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_t\|_2 \end{aligned} \tag{65}$$

By exploiting Lemma 13, (64) and (65), we have

$$
\begin{aligned}
\sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\tilde{\mathbf{x}}_{t,i})-f_{t,j}(\mathbf{x}^*)\right] &\overset{(64)}{\leq} \sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\hat{\mathbf{x}}_t)+G\|\hat{\mathbf{x}}_t-\tilde{\mathbf{x}}_{t,i}\|_2-f_{t,j}(\mathbf{x}^*)\right] \\
&\overset{(63),(65)}{\leq} \sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\tilde{\mathbf{x}}_{t,j})+G\|\hat{\mathbf{x}}_t-\tilde{\mathbf{x}}_{t,j}\|_2-f_{t,j}(\mathbf{x}^*)\right]+\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GT \\
&\overset{(63)}{\leq} \sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\tilde{\mathbf{x}}_{t,j})-f_{t,j}(\mathbf{x}^*)\right]+2\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GT \\
&\leq \sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left\langle\tilde{\nabla}_{t,j},\tilde{\mathbf{x}}_{t,j}-\mathbf{x}^*\right\rangle+2\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GT \\
&= \sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left[\left\langle\tilde{\nabla}_{t,j},\tilde{\mathbf{x}}_{t,j}-\hat{\mathbf{x}}_t\right\rangle+\left\langle\tilde{\nabla}_{t,j},\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle\right]+2\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GT \\
&\leq \sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left[G\|\tilde{\mathbf{x}}_{t,j}-\hat{\mathbf{x}}_t\|_2+\left\langle\tilde{\nabla}_{t,j},\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle\right]+2\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GT \\
&\overset{(63)}{\leq} \sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left\langle\tilde{\nabla}_{t,j},\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle+3\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GT.
\end{aligned}
\tag{66}
$$

Summing up both side from $j=1$ to n, we have

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{Regret}_i\right] &= \sum_{j=1}^{n}\sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\tilde{\mathbf{x}}_{t,i})-f_{t,j}(\mathbf{x}^*)\right] \\
&\leq \sum_{j=1}^{n}\sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left\langle\tilde{\nabla}_{t,j},\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle+3\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GTn \\
&\leq n\sum_{t=1}^{T}\mathbb{E}_{\xi_{[T]}}\left\langle\tilde{\nabla}_{t},\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle+3\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GTn,
\end{aligned}
\tag{67}
$$

in which $\tilde{\nabla}_t=\frac{1}{n}\sum_{j=1}^{n}\tilde{\nabla}_{t,j}$. $\tilde{\nabla}_{t,j}$ and $\tilde{\mathbf{x}}_{t,j}$ are defined in Algorithm 2.

Following the same proof framework of Theorem 1, we consider $\tilde{F}_t=\left\langle\tilde{\nabla}_t,\mathbf{x}\right\rangle$, where $\tilde{\nabla}_t$ is denoted as $\tilde{\nabla}_t=\frac{1}{n}\sum_{j=1}^{n}\tilde{\nabla}_{t,j}$. And we can derive the following lemma.

Lemma 14. *Define* $\mathbf{x}^*=\arg\min_{\mathbf{x}\in\mathcal{K}}\sum_{t=1}^{T}\sum_{j=1}^{n}f_{t,j}(\mathbf{x})$, $\tilde{\nabla}_t=\frac{1}{n}\sum_{j=1}^{n}\tilde{\nabla}_{t,j}$, $\tilde{F}_t(\mathbf{x})=\left\langle\tilde{\nabla}_t,\mathbf{x}\right\rangle$ *and* $\hat{\mathbf{x}}_t=\nabla h_\eta^*\left(-\tilde{\mathbf{z}}_{t-1}\right)$ *where* $\tilde{\mathbf{z}}_{t-1}=\frac{1}{n}\sum_{j=1}^{n}\tilde{\mathbf{z}}_{t-1,j}$, *then we have*

$$
\sum_{t=1}^{T}\tilde{F}_t(\hat{\mathbf{x}}_t)-\sum_{t=1}^{T}\tilde{F}_t(\mathbf{x}^*)\leq\frac{\eta dD}{2}G^2T+\frac{2D}{\eta}. \tag{68}
$$

By using Lemma 14, we can obtain that

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{Regret}_i\right] &\leq n\mathbb{E}\left[\sum_{t=1}^{T}\left\langle\tilde{\nabla}_t,\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle\right]+3\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GTn \\
&= n\mathbb{E}\left[\sum_{t=1}^{T}\left(\tilde{F}_t(\hat{\mathbf{x}}_t)-\tilde{F}_t(\mathbf{x}^*)\right)\right]+3\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GTn \\
&\leq n\left\{\frac{\eta dD}{2}G^2T+\frac{2D}{\eta}\right\}+3\left(\epsilon+\frac{2D}{\sqrt{m}}\right)GTn \\
&= \frac{2Dn}{\eta}+\frac{\eta dD}{2}G^2Tn+3\epsilon GTn+\frac{6DGTn}{\sqrt{m}} \\
&= \frac{2Dn}{\eta}+\eta dDG^2Tn\left(\frac{1}{2}+\frac{3\sqrt{n}}{1-\sigma_2(P)}\right)+\frac{6DGTn}{\sqrt{m}}.
\end{aligned}
\tag{69}
$$

Proof of smooth and convex losses

Lemma 15. *(Lemma 14 in Hazan and Minasyan (2020)) If the function $f:\mathcal{K}\longrightarrow\mathbb{R}$ is β-smooth, then we have*

$$\langle\nabla f(\mathbf{y})-\nabla f(\mathbf{x}),\mathbf{y}-\mathbf{x}\rangle\le\|\nabla f(\mathbf{y})-\nabla f(\mathbf{x})\|_2\cdot\|\mathbf{y}-\mathbf{x}\|_2\le\beta\|\mathbf{y}-\mathbf{x}\|_2^2, \tag{70}$$

which equals to

$$\langle\nabla f(\mathbf{y}),\mathbf{y}-\mathbf{x}\rangle\le\langle\nabla f(\mathbf{x}),\mathbf{y}-\mathbf{x}\rangle+\beta\|\mathbf{y}-\mathbf{x}\|_2^2. \tag{71}$$

Lemma 16. *Let $\tilde{\mathbf{x}}_{t,i}$ be defined as that in Algorithm 2 and define $\hat{\mathbf{x}}_t=\nabla h_\eta^*\left(-\tilde{\mathbf{z}}_{t-1}\right)$ where $\tilde{\mathbf{z}}_{t-1}=\frac{1}{n}\sum_{j=1}^n\tilde{\mathbf{z}}_{t-1,j}$, then we have*

$$\mathbb{E}_{\xi_{[T]}}\left[\langle\nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}),\tilde{\mathbf{x}}_{t,i}-\hat{\mathbf{x}}_t\rangle\right]\le\epsilon G+\frac{4\beta D^2}{m}, \tag{72}$$

$$\mathbb{E}_{\xi_{[T]}}\left[\langle\nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}),\tilde{\mathbf{x}}_{t,j}-\hat{\mathbf{x}}_t\rangle\right]\le\epsilon G+\frac{4\beta D^2}{m}, \tag{73}$$

$$\mathbb{E}_{\xi_{[T]}}\left[\langle\nabla f_{t,j}(\hat{\mathbf{x}}_t),\hat{\mathbf{x}}_t-\tilde{\mathbf{x}}_{t,j}\rangle\right]\le\epsilon G, \tag{74}$$

where $\epsilon=\eta dD\frac{\sqrt{n}G}{1-\sigma_2(P)}$.

Using the convexity of local loss functions $f_{t,j}$, triangle inequality and Lemma 16, we have

$$\begin{aligned}\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\tilde{\mathbf{x}}_{t,i})-f_{t,j}(\mathbf{x}^*)\right]&\le\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\hat{\mathbf{x}}_t)+\langle\nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}),\tilde{\mathbf{x}}_{t,i}-\hat{\mathbf{x}}_t\rangle-f_{t,j}(\mathbf{x}^*)\right]\\&\overset{(72)}{\le}\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\tilde{\mathbf{x}}_{t,j})+\langle\nabla f_{t,j}(\hat{\mathbf{x}}_t),\hat{\mathbf{x}}_t-\tilde{\mathbf{x}}_{t,j}\rangle-f_{t,j}(\mathbf{x}^*)\right]+\epsilon GT+\frac{4\beta D^2T}{m}\\&\overset{(74)}{\le}\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\tilde{\mathbf{x}}_{t,j})-f_{t,j}(\mathbf{x}^*)\right]+2\epsilon GT+\frac{4\beta D^2T}{m}\\&\le\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\langle\nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}),\tilde{\mathbf{x}}_{t,j}-\mathbf{x}^*\rangle+2\epsilon GT+\frac{4\beta D^2T}{m}\\&=\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\left[\langle\nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}),\tilde{\mathbf{x}}_{t,j}-\hat{\mathbf{x}}_t\rangle+\langle\nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}),\hat{\mathbf{x}}_t-\mathbf{x}^*\rangle\right]+2\epsilon GT+\frac{4\beta D^2T}{m}\\&\overset{(73)}{\le}\mathbb{E}_{\xi_{[T]}}\sum_{t=1}^T\left\langle\tilde{\nabla}_{t,j},\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle+3\epsilon GT+\frac{8\beta D^2T}{m},\end{aligned} \tag{75}$$

where $\tilde{\nabla}_{t,j}=\nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j})$.

Summing up both side from $j=1$ to n, we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{Regret}_i\right]&=\sum_{j=1}^n\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\left[f_{t,j}(\tilde{\mathbf{x}}_{t,i})-f_{t,j}(\mathbf{x}^*)\right]\\&\le\sum_{j=1}^n\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\left\langle\tilde{\nabla}_{t,j},\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle+3\epsilon GTn+\frac{8\beta D^2Tn}{m}\\&\le n\sum_{t=1}^T\mathbb{E}_{\xi_{[T]}}\left\langle\tilde{\nabla}_t,\hat{\mathbf{x}}_t-\mathbf{x}^*\right\rangle+3\epsilon GTn+\frac{8\beta D^2Tn}{m},\end{aligned} \tag{76}$$

in which $\tilde{\nabla}_t=\frac{1}{n}\sum_{j=1}^n\tilde{\nabla}_{t,j}$ and $\epsilon=\eta dD\frac{\sqrt{n}G}{1-\sigma_2(P)}$.

Therefore, we can upper bound the the expected regret as following

$$
\begin{aligned}
\mathbb{E}\left[\text{Regret}_i\right] &\leq n\mathbb{E}\left[\sum_{t=1}^{T}\left\langle\tilde{\nabla}_t, \hat{\mathbf{x}}_t - \mathbf{x}^*\right\rangle\right] + 3\epsilon GTn + \frac{8\beta D^2 Tn}{m} \\
&\leq n\left\{\frac{\eta dD}{2}G^2 T + \frac{2D}{\eta}\right\} + 3\epsilon GTn + \frac{8\beta D^2 Tn}{m} \\
&= \frac{2Dn}{\eta} + \frac{\eta dD}{2}G^2 Tn + 3\epsilon GTn + \frac{8\beta D^2 Tn}{m} \\
&= \frac{2Dn}{\eta} + \eta dDG^2 Tn\left(\frac{1}{2} + \frac{3\sqrt{n}}{1-\sigma_2(P)}\right) + \frac{8\beta D^2 Tn}{m},
\end{aligned} \tag{77}
$$

where the second inequality is due to Lemma 14.

Proof of Lemma 12

Let $\check{\mathbf{x}}_{t,i} = \nabla h_\eta^*\left(-\tilde{\mathbf{z}}_{t-1,i}\right)$ and $\hat{\mathbf{x}}_t = \nabla h_\eta^*\left(-\tilde{\mathbf{z}}_{t-1}\right)$, where $\tilde{\mathbf{z}}_{t-1} = \frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbf{z}}_{t-1,i}$ and $\tilde{\mathbf{z}}_{t-1,i}$ is defined in Algorithm 2. Then we have

$$
\begin{aligned}
\left\|\check{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\right\|_2 &= \left\|\nabla h_\eta^*\left(-\tilde{\mathbf{z}}_{t-1,i}\right) - \nabla h_\eta^*\left(-\tilde{\mathbf{z}}_{t-1}\right)\right\|_2 \\
&\overset{(40)}{\leq} \eta dD\left\|\tilde{\mathbf{z}}_{t-1} - \tilde{\mathbf{z}}_{t-1,i}\right\|_2 \\
&\overset{(41)}{\leq} \eta dD\frac{\sqrt{n}G}{1-\eta_2(P)},
\end{aligned} \tag{78}
$$

where the first inequality is due to the smoothness of $h_\eta^*(\mathbf{y})$ and the second inequality is due to Lemma 7.

Proof of Lemma 13

To prove Lemma 13, we define the following auxiliary variable,

$$
\check{\mathbf{x}}_{t,i} = \nabla h_\eta^*\left(-\tilde{\mathbf{z}}_{t-1,i}\right) = \mathbb{E}_{\mathbf{v}_t\sim\mathbb{B}}\left[\mathcal{O}_{\mathcal{K}}\left(-\tilde{\mathbf{z}}_{t-1,i} + \frac{\mathbf{v}_t}{\eta}\right)\right]. \tag{79}
$$

Using triangle inequality, it is easy to obtain that

$$
\|\hat{\mathbf{x}}_t - \tilde{\mathbf{x}}_{t,i}\|_2 \leq \|\hat{\mathbf{x}}_t - \check{\mathbf{x}}_{t,i}\|_2 + \|\check{\mathbf{x}}_{t,i} - \tilde{\mathbf{x}}_{t,i}\|_2. \tag{80}
$$

By using Lemma 12, we have

$$
\left\|\hat{\mathbf{x}}_t - \check{\mathbf{x}}_{t,i}\right\|_2 \overset{(78)}{\leq} \eta dD\frac{\sqrt{n}G}{1-\sigma_2(P)}. \tag{81}
$$

We know $\tilde{\mathbf{x}}_{t,i}$ is the unbiased estimation of $\check{\mathbf{x}}_{t,i}$. Following Hazan and Minasyan (2020), we take expectation over all randomness $\xi_{[T],i}$ with the reverse order $\xi_{T,i}, \cdots, \xi_{1,i}$. It is worth of attention that $\check{\mathbf{x}}_{t,i}$ is deterministic on $\xi_{t,i}$ given $\xi_{[t-1],i}$. Hence, we have

$$
\mathbb{E}_{\xi_{[T],i}}\left[\left\|\check{\mathbf{x}}_{t,i} - \tilde{\mathbf{x}}_{t,i}\right\|_2\right] = \mathbb{E}_{\xi_{[t],i}}\left[\left\|\check{\mathbf{x}}_{t,i} - \tilde{\mathbf{x}}_{t,i}\right\|_2\right] = \mathbb{E}_{\xi_{[t-1],i}}\left[\mathbb{E}_{\xi_{t,i}}\left[\|\check{\mathbf{x}}_{t,i} - \tilde{\mathbf{x}}_{t,i}\|_2 \,|\xi_{[t-1],i}\right]\right] \leq \frac{2D}{\sqrt{m}}. \tag{82}
$$

The inequality is due to Lemma 11. Because of $\{\xi_{t,1}, \cdots, \xi_{t,n}\}$ i.i.d., we have

$$
\mathbb{E}_{\xi_{[T]}}\left[\left\|\check{\mathbf{x}}_{t,i} - \tilde{\mathbf{x}}_{t,i}\right\|_2\right] \leq \frac{2D}{\sqrt{m}}. \tag{83}
$$

Therefore, by summing up above inequalities, we have

$$
\begin{aligned}
\mathbb{E}_{\xi_{[T]}}\left[\|\hat{\mathbf{x}}_t - \check{\mathbf{x}}_{t,i}\|_2\right] &\leq \mathbb{E}_{\xi_{[T]}}\left[\|\hat{\mathbf{x}}_t - \check{\mathbf{x}}_{t,i}\|_2 + \|\check{\mathbf{x}}_{t,i} - \tilde{\mathbf{x}}_{t,i}\|_2\right] \\
&\leq \eta dD\frac{\sqrt{n}G}{1-\sigma_2(P)} + \frac{2D}{\sqrt{m}} = \epsilon + \frac{2D}{\sqrt{m}},
\end{aligned} \tag{84}
$$

where $\epsilon = \eta dD\frac{\sqrt{n}G}{1-\sigma_2(P)}$.

Proof of Lemma 14

Following the same derivation, we define $\tilde{\lambda}_r^t (r = 1, \cdots, T)$ as

$$\tilde{\lambda}_r^t = \begin{cases} \tilde{\nabla}_r, & \text{if } r \leq t; \\ 0, & \text{if } r > t. \end{cases} \tag{85}$$

and consider the difference between $D(\tilde{\lambda}_1^t, \cdots, \tilde{\lambda}_T^t)$ and $D(\tilde{\lambda}_1^{t-1}, \cdots, \tilde{\lambda}_T^{t-1})$

$$\begin{aligned} \tilde{\Delta}_t &= D(\tilde{\lambda}_1^t, \cdots, \tilde{\lambda}_T^t) - D(\tilde{\lambda}_1^{t-1}, \cdots, \tilde{\lambda}_T^{t-1}) \\ &= D(\tilde{\nabla}_1, \cdots, \tilde{\nabla}_{t-1}, \tilde{\nabla}_t, \cdots, 0) - D(\tilde{\nabla}_1, \cdots, \tilde{\nabla}_{t-1}, 0, \cdots, 0) \\ &\overset{(39)}{\geq} \left\langle \tilde{\nabla}_t, \nabla h_\eta^* \left(-\tilde{\nabla}_{1:t-1} \right) \right\rangle - \tilde{F}_t^*(\tilde{\nabla}_t) - \frac{\eta d D}{2} G^2 + \tilde{F}_t^*(0) \\ &= \left\langle \tilde{\nabla}_t, \hat{\mathbf{x}}_t \right\rangle - \tilde{F}_t^*(\tilde{\nabla}_t) - \frac{\eta d D}{2} G^2 + \tilde{F}_t^*(0) \\ &= \tilde{F}_t(\hat{\mathbf{x}}_t) - \frac{\eta d D}{2} G^2 + \tilde{F}_t^*(0), \end{aligned} \tag{86}$$

where the first inequality is due to the smoothness of $h_\eta^*(\mathbf{y})$, the third equality is due to $\hat{\mathbf{x}}_t = \nabla h_\eta^* (-\tilde{\mathbf{z}}_{t-1}) = \nabla h_\eta^* \left(-\tilde{\nabla}_{1:t-1} \right)$ (Lemma 10) and the last equality is due to $\tilde{F}_t^*(\tilde{\nabla}_t) = \left\langle \tilde{\nabla}_t, \hat{\mathbf{x}}_t \right\rangle - \tilde{F}_t(\hat{\mathbf{x}}_t) = 0$ for the linear function $\tilde{F}_t(\mathbf{x}) = \left\langle \tilde{\nabla}_t, \mathbf{x} \right\rangle$. Then, following the similar derivation of Theorem 1, it is easy to obtain that

$$\sum_{t=1}^T \left\langle \tilde{\nabla}_t, \hat{\mathbf{x}}_t - \mathbf{x}^* \right\rangle = \sum_{t=1}^T \tilde{F}_t(\hat{\mathbf{x}}_t) - \sum_{t=1}^T \tilde{F}_t(\mathbf{x}^*) \leq \sum_{t=1}^T \tilde{F}_t(\hat{\mathbf{x}}_t) - \min_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T \tilde{F}_t(\mathbf{x}) \leq \frac{\eta d D}{2} G^2 T + \frac{2D}{\eta}. \tag{87}$$

Proof of Lemma 16

To prove Lemma 16, we define the following auxiliary variable,

$$\check{\mathbf{x}}_{t,i} = \nabla h_\eta^* (-\tilde{\mathbf{z}}_{t-1,i}). \tag{88}$$

proof of (72)
Using triangle inequality and Lemma 12, we have

$$\begin{aligned} \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t \rangle &= \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle + \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \check{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t \rangle \\ &\leq \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle + G \|\check{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\|_2 \\ &\overset{(78)}{\leq} \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle + \epsilon G, \end{aligned} \tag{89}$$

where $\epsilon = \eta d D \frac{\sqrt{n} G}{1 - \sigma_2(P)}$.

Now, proceed to bound the first term. By using Lemma 15, the first term can be rewritten as

$$\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle \leq \langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle + \beta \|\tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\|_2^2. \tag{90}$$

Moreover, $\tilde{\mathbf{x}}_{t,i}$ is is the unbiased estimation of $\check{\mathbf{x}}_{t,i}$ and $\nabla f_{t,j}(\check{\mathbf{x}}_{t,i})$ is independent of $\tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}$ with respect to $\xi_{t,i}$ condition on $\xi_{[t-1],i}$. Following Hazan and Minasyan (2020), we take expectation over all randomness $\xi_{[T],i}$ with the reverse order $\xi_{T,i}, \cdots, \xi_{1,i}$

$$\begin{aligned} \mathbb{E}_{\xi_{[T],i}} [\langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle] &= \mathbb{E}_{\xi_{[t],i}} [\langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle] \\ &= \mathbb{E}_{\xi_{[t-1],i}} \left[\mathbb{E}_{\xi_{t,i}} \left[\langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle | \xi_{[t-1],i} \right] \right] = 0. \end{aligned} \tag{91}$$

So combining with Lemma 11 and Lemma 15, we have

$$\begin{aligned} \mathbb{E}_{\xi_{[T],i}} [\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle] &= \mathbb{E}_{\xi_{[t],i}} [\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle] \\ &\leq \mathbb{E}_{\xi_{[t],i}} \left[\langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle + \beta \|\tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\|_2^2 \right] \leq \frac{4\beta D^2}{m}. \end{aligned} \tag{92}$$

Because of $\{\xi_{t,1}, \cdots, \xi_{t,n}\}$ i.i.d., we have

$$\mathbb{E}_{\xi_{[T]}} [\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i} \rangle] \leq \frac{4\beta D^2}{m}. \tag{93}$$

After bounding the first term of (89), we have

$$\mathbb{E}_{\xi_{[T]}}\left[\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\rangle\right] \le \frac{4\beta D^2}{m} + \epsilon G. \tag{94}$$

proof of (73)

$$\begin{aligned}\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_t\rangle &= \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle + \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \check{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_t\rangle \\ &\le \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle + G\|\check{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_t\|_2 \\ &\overset{(78)}{\le} \langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle + \epsilon G.\end{aligned} \tag{95}$$

The last inequality is because of Lemma 12. Also, by using Lemma 15, we have

$$\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle \le \langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle + \beta\|\tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\|_2^2. \tag{96}$$

For the same reason that $\tilde{\mathbf{x}}_{t,j}$ is is the unbiased estimation of $\check{\mathbf{x}}_{t,j}$ and $\nabla f_{t,j}(\check{\mathbf{x}}_{t,j})$ is independent of $\tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}$ with respect to $\xi_{t,j}$ condition on $\xi_{[t-1],j}$. So combining with Lemma 11 and Lemma 15, we take expectation over $\xi_{[T],j}$ with the reverse order $\xi_{T,j}, \cdots, \xi_{1,j}$:

$$\begin{aligned}\mathbb{E}_{\xi_{[T],j}}\left[\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle\right] &= \mathbb{E}_{\xi_{[t],j}}\left[\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle\right] \\ &\le \mathbb{E}_{\xi_{[t],j}}\left[\langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle + \beta\|\tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\|_2^2\right] \le \frac{4\beta D^2}{m}.\end{aligned} \tag{97}$$

Because of $\{\xi_{t,1}, \cdots, \xi_{t,n}\}$ i.i.d., we have

$$\mathbb{E}_{\xi_{[T]}}\left[\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle\right] \le \frac{4\beta D^2}{m}. \tag{98}$$

So $\mathbb{E}_{\xi_{[T]}}\left[\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_t\rangle\right]$ is upper bounded by

$$\mathbb{E}_{\xi_{[T]}}\left[\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_t\rangle\right] \le \frac{4\beta D^2}{m} + \epsilon G. \tag{99}$$

proof of (74)

$$\begin{aligned}\langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \hat{\mathbf{x}}_t - \tilde{\mathbf{x}}_{t,j}\rangle &= \langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \hat{\mathbf{x}}_t - \check{\mathbf{x}}_{t,j}\rangle + \langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \check{\mathbf{x}}_{t,j} - \tilde{\mathbf{x}}_{t,j}\rangle \\ &\le G\|\hat{\mathbf{x}}_t - \check{\mathbf{x}}_{t,j}\|_2 + \langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \check{\mathbf{x}}_{t,j} - \tilde{\mathbf{x}}_{t,j}\rangle \\ &\overset{(78)}{\le} \epsilon G + \langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \check{\mathbf{x}}_{t,j} - \tilde{\mathbf{x}}_{t,j}\rangle,\end{aligned} \tag{100}$$

where $\epsilon = \eta d D \frac{\sqrt{n}G}{1-\sigma_2(P)}$.

Also, $\tilde{\mathbf{x}}_{t,j}$ is is the unbiased estimation of $\check{\mathbf{x}}_{t,j}$ and $\nabla f_{t,j}(\hat{\mathbf{x}}_t)$ is independent on $\check{\mathbf{x}}_{t,j} - \tilde{\mathbf{x}}_{t,j}$ with respect to $\xi_{t,j}$ when condition on $\xi_{[t-1],j}$. So we take expectation over $\xi_{[T],j}$ with the reverse order $\xi_{T,j}, \cdots, \xi_{1,j}$:

$$\begin{aligned}\mathbb{E}_{\xi_{[T],j}}\left[\langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \check{\mathbf{x}}_{t,j} - \tilde{\mathbf{x}}_{t,j}\rangle\right] &= \mathbb{E}_{\xi_{[t],j}}\left[\langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \check{\mathbf{x}}_{t,j} - \tilde{\mathbf{x}}_{t,j}\rangle\right] \\ &= \mathbb{E}_{\xi_{[t-1],j}}\left[\mathbb{E}_{\xi_{t,j}}\left[\langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \check{\mathbf{x}}_{t,j} - \tilde{\mathbf{x}}_{t,j}\rangle \,|\xi_{[t-1],j}\right]\right] = 0.\end{aligned} \tag{101}$$

Because of $\{\xi_{t,1}, \cdots, \xi_{t,n}\}$ i.i.d., we have

$$\mathbb{E}_{\xi_{[T]}}\left[\langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \check{\mathbf{x}}_{t,j} - \tilde{\mathbf{x}}_{t,j}\rangle\right] = 0. \tag{102}$$

Therefore, the upper bound of $\mathbb{E}_{\xi_{[T]}}\left[\langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \hat{\mathbf{x}}_t - \tilde{\mathbf{x}}_{t,j}\rangle\right]$ is

$$\mathbb{E}_{\xi_{[T]}}\left[\langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \hat{\mathbf{x}}_t - \tilde{\mathbf{x}}_{t,j}\rangle\right] \le \epsilon G. \tag{103}$$

Proof of Theorem 3

Proof of general convex losses

Lemma 17. *(Proposition 17 in (Hazan and Minasyan 2020)) Suppose* $\{\mathbf{s}_1, \cdots, \mathbf{s}_m\}$ *is martingale-difference sequence defined on* $\{\mathcal{F}_1, \cdots, \mathcal{F}_m\}$*. So* $\{\mathbf{s}_1, \cdots, \mathbf{s}_m\}$ *holds that* $\forall u \in [1, m],\ \mathbb{E}[\mathbf{s}_u|\mathcal{F}_{u-1}] = 0$ *and* $\exists c_u > 0,\ \|\mathbf{s}_u\|_2 \le c_u$*. Then for all* $r \ge 0$

$$\mathbf{Pr}\left(\left\|\sum_{u=1}^{m}\mathbf{s}_u\right\|_2 \ge r\right) \le 2\exp\left\{-\frac{r^2}{2\sum_{u=1}^{m}c_u^2}\right\}. \tag{104}$$

We first define the following auxiliary variable,

$$\check{\mathbf{x}}_{t,i} = \nabla h_\eta^* \left(-\tilde{\mathbf{z}}_{t-1,i}\right). \tag{105}$$

Then, $\tilde{\mathbf{x}}_{t,i} = \frac{1}{m}\sum_{u=1}^m \tilde{\mathbf{x}}_{t,i}^u$ is is the unbiased estimation of $\check{\mathbf{x}}_{t,i}$ Denote $\mathbf{s}_u = \frac{1}{m}\left(\tilde{\mathbf{x}}_{t,i}^u - \check{\mathbf{x}}_{t,i}\right)$ for learner i at round t, which is the martingale-difference sequence on $\{\mathcal{F}_1, \cdots, \mathcal{F}_m\} = \{\mathbf{v}_{t,i}^1, \cdots, \mathbf{v}_{t,i}^m\} = \xi_{t,i}$. Then, we have $\sum_{u=1}^m \mathbf{s}_u = \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}$ as well as $\mathbb{E}_{\mathbf{v}_{t,i}^u}[\mathbf{s}_u | \mathbf{v}_{t,i}^1, \cdots, \mathbf{v}_{t,i}^{u-1}] = 0$ due to the unbiased estimation and i.i.d. samples from an unit ball $\mathbb{B}$. According to Assumption 1, there is $\|\mathbf{s}_u\|_2 = \|\frac{1}{m}\left(\tilde{\mathbf{x}}_{t,i}^u - \check{\mathbf{x}}_{t,i}\right)\|_2 = \frac{\|\tilde{\mathbf{x}}_{t,i}^u - \check{\mathbf{x}}_{t,i}\|_2}{m} \le \frac{2D}{m} = c_t$. By Lemma 17, we can obtain that

$$\mathbf{Pr}_{\xi_{t,i}}\left(\left\|\frac{1}{m}\sum_{u=1}^m \left(\tilde{\mathbf{x}}_{t,i}^u - \check{\mathbf{x}}_{t,i}\right)\right\|_2 \ge r\right) \le 2\exp\left\{-\frac{r^2}{\frac{8D^2}{m}}\right\}. \tag{106}$$

For some $\delta > 0$, let $r = 2D\sqrt{\frac{2}{m}\ln\frac{2T}{\delta}}$ and there is

$$\mathbf{Pr}_{\xi_{t,i}}\left(\left\|\frac{1}{m}\sum_{u=1}^m \left(\tilde{\mathbf{x}}_{t,i}^u - \check{\mathbf{x}}_{t,i}\right)\right\|_2 \ge r\right) \le \frac{\delta}{T}. \tag{107}$$

Because of $\tilde{\mathbf{x}}_{t,i} = \frac{1}{m}\sum_{u=1}^m \tilde{\mathbf{x}}_{t,i}^u$ and $\{\xi_{t,1}, \cdots, \xi_{t,n}\}$ i.i.d., for the whole interval $[1, T]$ the union bound is

$$\mathbf{Pr}_{\xi_{[T]}}\left(\forall t \in [1,T],\ \left\|\tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\right\|_2 \ge r\right) \le \delta, \tag{108}$$

which also means

$$\mathbf{Pr}_{\xi_{[T]}}\left(\forall t \in [1,T],\ \left\|\tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\right\|_2 \le r\right) \ge 1-\delta. \tag{109}$$

Therefore, with at least $1-\delta$ probability, $\|\hat{\mathbf{x}}_t - \tilde{\mathbf{x}}_{t,i}\|_2$ is bounded as following

$$\|\hat{\mathbf{x}}_t - \tilde{\mathbf{x}}_{t,i}\|_2 \le \|\hat{\mathbf{x}}_t - \check{\mathbf{x}}_{t,i}\|_2 + \|\check{\mathbf{x}}_{t,i} - \tilde{\mathbf{x}}_{t,i}\|_2 \overset{(78),(109)}{\le} \epsilon + r, \tag{110}$$

where $r = 2D\sqrt{\frac{2}{m}\ln\frac{2T}{\delta}}$ and $\epsilon = \eta d D\frac{\sqrt{n}G}{1-\sigma_2(P)}$.

Following the same proof framework of Theorem 2, we have

$$\mathrm{Regret}_i \le n\left\{\sum_{t=1}^T \left\langle\tilde{\nabla}_t, \hat{\mathbf{x}}_t - \mathbf{x}^*\right\rangle\right\} + 3(\epsilon + r)GTn. \tag{111}$$

Using Lemma 14, with at least $1-\delta$ probability, Algorithm 2 guarantees

$$\begin{aligned}\mathrm{Regret}_i &\le n\left\{\sum_{t=1}^T \left\langle\tilde{\nabla}_t, \hat{\mathbf{x}}_t - \mathbf{x}^*\right\rangle\right\} + 3(\epsilon + r)GTn \\ &\le n\left\{\frac{\eta d D}{2}G^2 T + \frac{2D}{\eta}\right\} + 3(\epsilon + r)GTn \\ &= \frac{2Dn}{\eta} + \frac{\eta d D}{2}G^2 Tn + 3(\epsilon + r)GTn \\ &= \frac{2Dn}{\eta} + \eta d D G^2 Tn\left(\frac{1}{2} + \frac{3\sqrt{n}}{1-\sigma_2(P)}\right) + 3rGTn\end{aligned} \tag{112}$$

where $r = 2D\sqrt{\frac{2}{m}\ln\frac{2T}{\delta}}$.

Proof of smooth and convex losses

Denote $\mathbf{g}_t$ satisfies $\|\mathbf{g}_t\|_2 \le G$ and $\mathbb{E}[\langle\mathbf{g}_t, \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle | \xi_{1,j}, \cdots, \xi_{t-1,j}] = 0$. Let $\mathbf{s}_t = \langle\mathbf{g}_t, \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle$ for learner j at round t, which is the martingale-difference sequence on $\{\mathcal{F}_1, \cdots, \mathcal{F}_T\} = \{\xi_{1,j}, \cdots, \xi_{T,j}\}$. Because $\mathbb{E}[\mathbf{s}_t | \xi_{1,j}, \cdots, \xi_{t-1,j}] = 0$ and $\|\mathbf{s}_t\|_2 = \|\langle\mathbf{g}_t, \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle\|_2 \le G\|\tilde{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_{t,j}\|_2 \le 2GD = c_t$. By Lemma 17, it can be obtained that

$$\mathbf{Pr}_{\xi_{[T],j}}\left(\left|\sum_{t=1}^T \langle\mathbf{g}_t, \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle\right| \ge r'\right) \le 2\exp\left\{-\frac{r'^2}{8G^2D^2T}\right\} = \delta' \tag{113}$$

As it is mentioned in the previous section, for some $\delta > 0$, there is $\|\tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\|_2 \le r$, in which $r = 2D\sqrt{\frac{2}{m}\ln\frac{2T}{\delta}}$. Now, let $\delta' = \frac{\delta}{2}$ and $r' = 2DG\sqrt{2T\ln\frac{2}{\delta'}} = 2DG\sqrt{2T\ln\frac{4}{\delta}}$. Then, there is

$$\mathbf{Pr}_{\xi_{[T],j}}\left(\left|\sum_{t=1}^{T}\langle \mathbf{g}_t, \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle\right| \ge r'\right) \le \delta'. \tag{114}$$

Because of $\{\xi_{t,1}, \cdots, \xi_{t,n}\}$ i.i.d., for the whole interval $[1, T]$ the union bound is

$$\mathbf{Pr}_{\xi_{[T]}}\left(\left|\sum_{t=1}^{T}\langle \mathbf{g}_t, \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle\right| \ge r'\right) \le \delta', \tag{115}$$

which also means

$$\mathbf{Pr}_{\xi_{[T]}}\left(\left|\sum_{t=1}^{T}\langle \mathbf{g}_t, \tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\rangle\right| \le r'\right) \ge 1 - \delta'. \tag{116}$$

Following the same proof framework of Lemma 16, we can derive

$$\begin{aligned}\sum_{t=1}^{T}\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\rangle &= \sum_{t=1}^{T}\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\rangle + \sum_{t=1}^{T}\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \check{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\rangle \\ &\overset{(78)}{\le} \sum_{t=1}^{T}\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\rangle + \epsilon GT \\ &\overset{(71)}{\le} \sum_{t=1}^{T}\langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\rangle + \beta\sum_{t=1}^{T}\|\tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\|_2^2 + \epsilon GT \\ &\overset{(116),(109)}{\le} r' + \beta r^2 T + \epsilon GT,\end{aligned} \tag{117}$$

where $r = 2D\sqrt{\frac{2}{m}\ln\frac{2T}{\delta}}$, $r' = 2DG\sqrt{2T\ln\frac{4}{\delta}}$ and $\epsilon = \eta dD\frac{\sqrt{n}G}{1-\sigma_2(P)}$. The first inequality is due to Lemma 12. The second inequality is due to Lemma 15. The last inequality is because that $\nabla f_{t,j}(\check{\mathbf{x}}_{t,i})$ is independent of $\tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}$ condition on $\xi_{[t-1],i}$ and satisfies $\mathbb{E}[\langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\rangle\,|\xi_{1,i}, \cdots, \xi_{t-1,i}] = 0$. Hence, $\sum_{t=1}^{T}\langle \nabla f_{t,j}(\check{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \check{\mathbf{x}}_{t,i}\rangle \le r'$ with at least $1-\delta$ probability. Meanwhile, we also have $\|\tilde{\mathbf{x}}_{t,j} - \check{\mathbf{x}}_{t,j}\|_2 \le r$.

By the same way, we can obtain that

$$\sum_{t=1}^{T}\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,i}), \tilde{\mathbf{x}}_{t,i} - \hat{\mathbf{x}}_t\rangle \le \epsilon GT + \beta r^2 T + r', \tag{118}$$

$$\sum_{t=1}^{T}\langle \nabla f_{t,j}(\tilde{\mathbf{x}}_{t,j}), \tilde{\mathbf{x}}_{t,j} - \hat{\mathbf{x}}_t\rangle \le \epsilon GT + \beta r^2 T + r', \tag{119}$$

$$\sum_{t=1}^{T}\langle \nabla f_{t,j}(\hat{\mathbf{x}}_t), \hat{\mathbf{x}}_t - \tilde{\mathbf{x}}_{t,j}\rangle \le \epsilon GT + r'. \tag{120}$$

Therefore, we can also obtain

$$\text{Regret}_i \le n\left\{\sum_{t=1}^{T}\left\langle \tilde{\nabla}_t, \hat{\mathbf{x}}_t - \mathbf{x}^*\right\rangle\right\} + 3\epsilon GTn + 3r'n + 2\beta r^2 Tn. \tag{121}$$

Using Lemma 14, with at least $1-\delta$ probability, Algorithm 2 guarantees

$$\begin{aligned}\text{Regret}_i &\le n\left\{\sum_{t=1}^{T}\left\langle \tilde{\nabla}_t, \hat{\mathbf{x}}_t - \mathbf{x}^*\right\rangle\right\} + 3\epsilon GTn + 3r'n + 2\beta r^2 Tn \\ &\le n\left\{\frac{\eta dD}{2}G^2T + \frac{2D}{\eta}\right\} + 3\epsilon GTn + 3r'n + 2\beta r^2 Tn \\ &= \frac{2Dn}{\eta} + \eta dDG^2Tn\left(\frac{1}{2} + \frac{3\sqrt{n}}{1-\sigma_2(P)}\right) + 3r'n + 2\beta r^2 Tn,\end{aligned} \tag{122}$$

where $r' = 2DG\sqrt{2T\ln\frac{4}{\delta}}$ and $r = 2D\sqrt{\frac{2}{m}\ln\frac{2T}{\delta}}$.

Proof of Theorem 4

Algorithm 3 can be reduced to Algorithm 2 with new settings, e.g., the number of rounds $T' = T/k$ and the block losses $f'_{t',i} = \sum_{t=(t'-1)\cdot k+1}^{t'\cdot k} f_{t,i}$ in the reduced game. Here, we list some crucial changes.

- In Assumption 1, the domain set in the reduced game is upper bounded by $D' = D$.
- In Assumption 2, for the block loss function $f'_{t',i} = \sum_{t=(t'-1)\cdot k+1}^{t'\cdot k} f_{t,i}$, the Lipschitz constant in the reduced game is $G' = k \cdot G$.
- If $f_{t,i}$ is β-smooth, then the block loss function $f'_{t',i} = \sum_{t=(t'-1)\cdot k+1}^{t'\cdot k} f_{t,i}$ is $(k \cdot \beta)$-smooth.

Proof of general convex losses

After reduction, Algorithm 3 also guarantees Theorem 2 for general convex losses.

$$\begin{aligned}\mathbb{E}\left[\text{Regret}_i\right] &\leq \frac{2D'n}{\eta} + \eta dDG'^2T'nL + \frac{6D'G'T'n}{\sqrt{m}} \\ &= \frac{2Dn}{\eta} + \eta dD(k\cdot G)^2T'nL + \frac{6D(k\cdot G)T'n}{\sqrt{m}},\end{aligned} \tag{123}$$

where $L = \frac{1}{2} + \frac{3\sqrt{n}}{1-\sigma_2(P)}$. With $\eta = \frac{1}{kG}\sqrt{\frac{2}{dLT'}}$ and $m = k$

$$\mathbb{E}\left[\text{Regret}_i\right] \leq 2kDGn\sqrt{2dLT'} + 6D\sqrt{k}GT'n. \tag{124}$$

Let $T' = T^{\frac{1}{2}}$ and $k = T^{\frac{1}{2}}$

$$\mathbb{E}\left[\text{Regret}_i\right] \leq nDG\left(2\sqrt{2dL} + 6\right)T^{\frac{3}{4}} = \mathcal{O}\left(T^{\frac{3}{4}}\right). \tag{125}$$

Proof of smooth and convex losses

After reduction, Algorithm 3 also guarantees Theorem 2 for smooth and convex losses.

$$\begin{aligned}\mathbb{E}\left[\text{Regret}_i\right] &\leq \frac{2D'n}{\eta} + \eta dDG'^2T'nL + \frac{8\beta'D'^2T'n}{m} \\ &= \frac{2Dn}{\eta} + \eta dD(k\cdot G)^2T'nL + \frac{8(k\cdot\beta)D^2T'n}{m},\end{aligned} \tag{126}$$

where $L = \frac{1}{2} + \frac{3\sqrt{n}}{1-\sigma_2(P)}$. With $\eta = \frac{1}{kG}\sqrt{\frac{2}{dLT'}}$ and $m = k$

$$\mathbb{E}\left[\text{Regret}_i\right] \leq 2kDGn\sqrt{2dLT'} + 8\beta D^2T'n. \tag{127}$$

Let $T' = T^{\frac{2}{3}}$ and $k = T^{\frac{1}{3}}$

$$\mathbb{E}\left[\text{Regret}_i\right] \leq nD\left(2G\sqrt{2dL} + 8\beta D\right)T^{\frac{2}{3}} = \mathcal{O}\left(T^{\frac{2}{3}}\right). \tag{128}$$

Proof of Theorem 5

Following the same proof framework as Theorem 4, we list some crucial changes after reduction.

- In Assumption 1, the domain set in the reduced game is upper bounded by $D' = D$.
- In Assumption 2, for the block loss function $f'_{t',i} = \sum_{t=(t'-1)\cdot k+1}^{t'\cdot k} f_{t,i}$, the Lipschitz constant in the reduced game is $G' = k \cdot G$.
- If $f_{t,i}$ is β-smooth, then the block loss function $f'_{t',i} = \sum_{t=(t'-1)\cdot k+1}^{t'\cdot k} f_{t,i}$ is $(k \cdot \beta)$-smooth.

Proof of general convex losses

After reduction, Algorithm 3 also guarantees Theorem 3 with $1-\delta$ probability for general convex losses.

$$\begin{aligned}\text{Regret}_i &\leq \frac{2D'n}{\eta} + \eta dD'G'^2T'nL + 3rG'T'n \\ &= \frac{2D'n}{\eta} + \eta dD'G'^2T'nL + 6D'G'T'n\sqrt{\frac{2}{m}\ln\frac{2T'}{\delta}} \\ &= \frac{2Dn}{\eta} + \eta dD(k\cdot G)^2T'nL + 6D(k\cdot G)T'n\sqrt{\frac{2}{m}\ln\frac{2T'}{\delta}}\end{aligned} \tag{129}$$

where $L = \frac{1}{2} + \frac{3\sqrt{n}}{1-\sigma_2(P)}$ and $r = 2D'\sqrt{\frac{2}{m}\ln\frac{2T'}{\delta}}$. With $\eta = \frac{1}{kG}\sqrt{\frac{2}{dLT'}}$ and $m = k$

$$\text{Regret}_i \leq 2kDGn\sqrt{2dLT'} + 6DGT'n\sqrt{2k\ln\frac{2T'}{\delta}}. \tag{130}$$

Let $T' = T^{\frac{1}{2}}$ and $k = T^{\frac{1}{2}}$

$$\text{Regret}_i \leq DGn\left(2\sqrt{2dL} + 6\sqrt{2\ln\frac{2T^{1/2}}{\delta}}\right)T^{\frac{3}{4}} = \tilde{\mathcal{O}}\left(T^{\frac{3}{4}}\ln\frac{1}{\delta}\right). \tag{131}$$

Proof of smooth and convex losses

After reduction, Algorithm 3 also guarantees Theorem 3 with $1-\delta$ probability for smooth and convex losses.

$$\begin{aligned}
\text{Regret}_i &\leq \frac{2D'n}{\eta} + \eta dD'G'^2T'nL + 3r'n + 2\beta' r^2T'n \\
&= \frac{2D'n}{\eta} + \eta dD'G'^2T'nL + 6D'G'n\sqrt{2T'\ln\frac{4}{\delta}} + \frac{16\beta' D'^2T'n}{m}\ln\frac{2T'}{\delta} \\
&= \frac{2Dn}{\eta} + \eta dD(k\cdot G)^2T'nL + 6D(k\cdot G)n\sqrt{2T'\ln\frac{4}{\delta}} + \frac{16(k\cdot\beta)D^2T'n}{m}\ln\frac{2T'}{\delta}
\end{aligned} \tag{132}$$

where $L = \frac{1}{2} + \frac{3\sqrt{n}}{1-\sigma_2(P)}$, $r = 2D'\sqrt{\frac{2}{m}\ln\frac{2T'}{\delta}}$ and $r' = 2D'G'\sqrt{2T'\ln\frac{4}{\delta}}$. With $\eta = \frac{1}{kG}\sqrt{\frac{2}{dLT'}}$ and $m = k$

$$\text{Regret}_i \leq 2kDGn\sqrt{2dLT'} + 6D(k\cdot G)n\sqrt{2T'\ln\frac{4}{\delta}} + 16\beta D^2T'n\ln\frac{2T'}{\delta} \tag{133}$$

Let $T' = T^{\frac{2}{3}}$ and $k = T^{\frac{1}{3}}$

$$\text{Regret}_i \leq Dn\left(2G\sqrt{2dL} + 6G\sqrt{2\ln\frac{4}{\delta}} + 16\beta D\ln\frac{2T^{2/3}}{\delta}\right)T^{\frac{2}{3}} = \tilde{\mathcal{O}}\left(T^{\frac{2}{3}}\ln\frac{1}{\delta}\right). \tag{134}$$